ISRAEL JOURNAL OF MATHEMATICS, Vol. 24, Nos. 3–4, 1976

AN R^d ANALOGUE OF VALENTINE'S THEOREM ON 3–CONVEX SETS

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ABSTRACT

This paper deals with an R^d analogue of a theorem of Valentine which states that a closed 3-convex set S in the plane is decomposable into 3 or fewer closed convex sets. In Valentine's proof, the points of local nonconvexity of S are treated as vertices of a polygon P contained in the kernel of S, yielding a decomposition of S into 2 or 3 convex sets, depending on whether P has an even or odd number of edges. Thus the decomposition actually depends on $c(P')$, the chromatic number of the polytope P' dual to P.

A natural analogue of this result is the following theorem: Let S be a closed subset of R^d, and let Q denote the set of points of local nonconvexity of S. We require that Q be contained in the kernel of S and that Q coincide with the set of points in the union of all the $(d-2)$-dimensional faces of some d-dimensional polytope P. Then S is decomposable into $c(P')$ closed convex sets.

Introduction

Let S be a subset of R^d. The set S is said to be 3-*convex* if and only if for every 3-member subset of S, at least one of the line segments determined by these points lies in S. A point q in S is called a *point of local convexity of* S if and only if there is some neighborhood N of q for which $N \cap S$ is convex. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Valentine [3] has proved that a closed 3-convex set S in the plane is decomposable into 3 or fewer closed convex sets, and in this paper, an attempt is made to obtain conditions under which an analogue of Valentine's result may be proved in R^d. A key construction in Valentine's proof involves the set of lnc points of S, which may be treated essentially as vertices of a polygon P contained in the kernel of S, yielding a three member decomposition of S when P has an odd number of edges, and a two member decomposition otherwise. Since for any closed 3-convex set S, the set of lnc points of S lies in $\ker S$, we replace Valentine's requirement that S be 3-convex with this weaker hypothesis.

Received June 22, 1974

The following notation will be employed: If P is a d-dimensional polytope, each facet F of P determines a hyperplane $H = \text{aff } F$, and throughout the paper, we will assume that $P \subseteq \text{cl}(H_1)$, where H_1, H_2 denote distinct open halfspaces corresponding to H. Moreover, we adopt the following familiar terminology: For any point x in R^d, we say that x is *beyond* F if x is in H_2 and that x is *beneath* F if x is in H_1.

As usual, conv S, aff S, cl S, int S, and ker S will denote the convex hull, affine hull, closure, interior, and kernel, respectively, for the set S.

THEOREM 1. *Let S be a closed subset of R^d and let Q denote the set of lnc points of S. If Q is contained in* ker S *and if Q coincides with the set of points in the union of all the $(d-2)$-dimensional faces of some d-dimensional convex polytope P, then S is decomposable into* $\text{c}(P')$ *closed convex sets, where* $\text{c}(P')$ *denotes the chromatic number of the graph of the polytope P' dual to P.*

PROOF. The proof will require several steps: First we show that every point of S lies beyond at most one facet of P, and for each facet F of P, the subset (wedge) W_F of S beyond F is convex. For facets F and G of P which do not intersect in a $(d-2)$-face, the set $W_F \cup W_G \cup P$ is convex. Finally, the chromatic decomposition for vertices of P' induces a partition among facets of P, with no associated facets intersecting in a $(d-2)$-face. This gives a natural decomposition of wedges of S into $\text{c}(P')$ convex sets.

We begin with the following lemma.

LEMMA 1. *Let F be any facet of P, V any set of vertices of P with $V \not\subseteq F$, and let $\mathcal{K}$ denote the collection of hyperplanes determined by facets K of* $\text{conv}(F \cup V)$ *with $K \neq F$. If x is a point of S beyond F, then* $x \in \cap\{\text{cl}(H_1) : H \text{ in } \mathcal{K}\}$.

PROOF OF LEMMA 1. Note that since $V \not\subseteq F$, the polytope $\text{conv}(F \cup V)$ is d-dimensional. Also, since $Q \subseteq \ker S$, it follows that $\text{conv}\, Q = P \subseteq \ker S$. Therefore $S = \text{cl}(\text{int}\, S)$, and since $\cap\{\text{cl}(H_1) : H \text{ in } \mathcal{K}\}$ is closed, without loss of generality we may assume that $x \in \text{int}\, S$ and that x is not in the affine hull of any d-member subset of the vertex set of P.

To prove the lemma, assume on the contrary that $x \notin \cap\{\text{cl}(H_1) : H \text{ in } \mathcal{K}\}$, to obtain a contradiction. Then there is some facet $K \neq F$ of conv $(F \cup V)$ with x beyond K. We show that K may be selected so that $K \cap F$ is a $(d-2)$-dimensional face of F.

If $\dim K \cap F \leqq d-3$, consider the polytope F as a subset of the $(d-1)$-dimensional space aff F. Let $\mathcal{A}_0$ denote the collection of all $(d-2)$-dimensional hyperplanes in aff F determined by facets A_0 of F. Clearly aff K cannot contain any point relatively interior to $\cap\{\mathrm{cl}(H_1): H \text{ in } \mathcal{A}_0\}$ since this intersection is exactly F. For each A_0 in $\mathcal{A}_0$, we select the corresponding facet A of $\mathrm{conv}(F \cup V)$ distinct from F and containing A_0, and define $\mathcal{A} = \{A : A_0 \text{ in } \mathcal{A}_0\}$. Then $\mathcal{A}$ is exactly the collection of facets of $\mathrm{conv}(F \cup V)$ which share a $(d-2)$-face with F. Moreover, every vertex of K lies in or beneath $A, A \in \mathcal{A} \cup \{F\}$, and since $\mathrm{aff}\, K$ contains no relative interior point of $\cap\{\mathrm{cl}(H_1): H \text{ in } \mathcal{A}_0\}$ in $\mathrm{aff}\, F$, the region

$$\cap\{(\mathrm{aff}\, A)_1 : A \text{ in } \mathcal{A}\} \cap (\mathrm{aff}\, F)_2 \cap (\mathrm{aff}\, K)_2$$

is empty. We are assuming that $x \notin \mathrm{aff}\, A$ for A in $\mathcal{A}$, so x must lie beyond some A in $\mathcal{A}$, and we may indeed choose K so that $K \cap F$ is a $(d-2)$-face of F.

Select $q \in \mathrm{relint}(K \cap F)$, the relative interior of the set $K \cap F$. (In case $d = 2$, then $K \cap F = \{q\}$.) Consider the ray $R(x, q)$ emanating from x through q. We assert that $R(x, q)$ contains points interior to $\mathrm{conv}(F \cup V)$: Let J be any facet of $\mathrm{conv}(F \cup V)$ to show that $R(x,q) \sim [x, q]$ contains an interval (q, r) beneath J. There are three possibilities to consider. Recall that by our previous assumption, $x \notin \mathrm{aff}\, J$. In case x is beyond J, then since q is either in or beneath J, $R(x, q) \sim [x, q]$ necessarily lies beneath J. If x is beneath J and q is, too, then certainly $R(x, q) \sim [x, q]$ contains some open interval (q, r) beneath J. The only remaining possibility is that x lie beneath J and q lie in J. In this event, J could not be F or K (since x is beyond both F and K). Furthermore, since $q \in F \cap K \cap J$, $F \cap K \cap J$ would be a nonempty face of $\mathrm{conv}(F \cup V)$, and since no three distinct facets may intersect in a $(d-2)$-face, $F \cap K \cap J$ would be a face of $\mathrm{conv}(F \cup V)$ of dimension $\leqq d-3$, and $3 \leqq d$. Then since $q \in F \cap K \cap J$, q could not belong to $\mathrm{relint}(F \cap K)$. Therefore, this case cannot occur. We conclude that $R(x, q) \sim [x, q]$ necessarily contains some interval (q, r) beneath J for every facet J of $\mathrm{conv}(F \cup V)$. Using the fact that there are finitely many such facets, there is some r_0 on $R(x, q) \sim [x, q]$ for which $(q, r_0) \subseteq \mathrm{int\,conv}(F \cup V) \subseteq \ker S$, and the assertion is proved.

If N is any neighborhood of x in S, then $\mathrm{conv}(N \cup \{r_0\})$ lies in S and contains q as an interior point, contradicting the fact that q is an lnc point of S. At last we have a contradiction, our assumption is false, and x indeed lies in $\cap\{\mathrm{cl}(H_1): H \text{ in } \mathcal{H}\}$, completing the proof of Lemma 1.

For each facet F of P, we define the *wedge* W_F determined by F to be the set of points of S which lie beyond F. Clearly every point of S is either in P or in some wedge, and by Lemma 1, every pair of distinct wedges are disjoint. We assert that each wedge of S is convex: Let F be any facet of P, $W = W_F$ the corresponding wedge with x, y in W. Select $p \in \mathrm{relint}\, F$. Then $p \in \ker S$, so

$[p, x] \cup [p, y] \subseteq S$. Since x and y are beyond F and $p \not\in Q$, there can be no lnc point of S in $\operatorname{conv}\{p, x, y\}$, and by a lemma of Valentine [4, cor. 1], $\operatorname{conv}\{p, x, y\} \subseteq S$. Thus $[x, y] \subseteq S$, and since $[x, y]$ is beyond $F, [x, y] \subseteq W$, the desired result.

To obtain the decomposition for S, one more lemma will be needed.

LEMMA 2. *Let F, G be facets of P with* $\dim(F \cap G) \leqq d - 3$, *and let W, U be the wedges of S determined by F, G respectively. Then $W \cup U \cup P$ is convex.*

PROOF OF LEMMA 2. Since P, W, and U are convex, it is sufficient to consider $p \in P, w \in W, u \in U$, to show that each of the corresponding segments is in $W \cap U \cup P$.

To see that $[w, u] \subseteq W \cup U \cup P$, let T denote the polytope $\operatorname{conv}(F \cup G)$, and let $\mathcal{J}$ denote the collection of hyperplanes determined by facets J of T, where $J \neq F, G$. Then by Lemma 1, each of w and u must lie in $\cap\{\operatorname{cl}(H_1): H \text{ in } \mathcal{J}\}$, and $[w, u] \subseteq \cap\{\operatorname{cl}(H_1): H \text{ in } \mathcal{J}\}$. Furthermore, since there is a member of $\mathcal{J}$ corresponding to every $(d-2)$-face of F, and w is beyond F while u is not, $[w, u]$ must intersect $F \subseteq \ker S$, and $[w, u] \subseteq S$. Then using the fact that $[w, u]$ intersects both F and G, it is easy to show that $[w, u] \subseteq W \cup U \cup P$.

Similarly, to show $[w, p] \subseteq W \cup U \cup P$, let $\mathcal{K}$ denote the collection of hyperplanes determined by facets K of P, where $K \neq F$. Repeating our earlier argument, $[w, p]$ intersects F, and $[w, p] \subseteq W \cup P$. By a parallel proof, $[u, p] \subseteq U \cup P$. Thus each of the segments lies in $W \cup U \cup P$ and $W \cup U \cup P$ is convex, finishing the proof of Lemma 2.

We obtain a decomposition for S in the following manner: Let P' denote the polytope dual to P, $\mathrm{c}(P')$ the chromatic number of the graph of P'. (The reader is referred to [1, p. 46, p. 212] and [2, p. 224] for the necessary definitions.) Let $\{\mathcal{V}_i : 1 \leqq i \leqq \mathrm{c}(P')\}$ be a chromatic decomposition for the vertices of P'. Recall that for v, w in $\mathcal{V}_i$, v and w are not joined by an edge of P'. Since there is a one-one correspondence between vertices of P' and facets of P, the decomposition $\{\mathcal{V}_i\}$ induces a partition $\{\mathcal{F}_i : 1 \leqq i \leqq \mathrm{c}(P')\}$ among facets of P having the property that for F, G in $\mathcal{F}_i$, F and G do not intersect in a $(d-2)$-face of P.

The partition $\{\mathcal{F}_i : 1 \leqq i \leqq \mathrm{c}(P')\}$ yields the required decomposition for S. For each $\mathcal{F}_i$, let $\mathcal{W}_i$ denote the collection of wedges of S determined by members of $\mathcal{F}_i$. Using Lemma 2, an easy induction shows that $S_i \equiv \cup\{W \cup P : W \text{ in } \mathcal{W}_i\}$ is convex, $1 \leqq i \leqq \mathrm{c}(P')$. Since $S = \cup\{S_i : 1 \leqq i \leqq \mathrm{c}(P')\}$, this finishes the proof of the theorem.

The author wishes to thank the referee for suggesting the following result.

THEOREM 2. *If P is a d-dimensional convex polytope, then there exists a compact set S having the following properties*:

1) *The set Q of lnc points of S is contained in* $\ker S$, *and Q is the union of all $(d-2)$-faces of P.*

2) *If S is decomposed into k closed convex sets, then $k \geqq \mathrm{c}(P')$, where $\mathrm{c}(P')$ is the chromatic number of the graph of the polytope P' dual to P.*

PROOF. For each facet F of P, define the wedge of F to be $\{x : x$ lies beyond F and beyond no other facet of $P\}$. Then S should consist of P and the wedges (or suitably large subsets of wedges) of all the facets of P.

Note that Theorem 2 implies the bound $\mathrm{c}(P')$ is best possible. Also notice that for $d = 2, \mathrm{c}(P')$ is either 2 or 3, paralleling results obtained by Valentine.

In conclusion, something should be said about the case in which Q is properly contained in the union of the $(d-2)$-faces of P. Without additional hypothesis, there is no k such that S is decomposable into k convex sets, as the following example reveals.

EXAMPLE 1. For k fixed, let $P = T_0$ be a square in R^2, v a vertex of T_0, and for $1 \leqq i \leqq k$, let T_i be a segment with $T_j \cap T_i = \{v\}, 0 \leqq j < i \leqq k$. If $S = \cup\{T_i : 0 \leqq i \leqq k\}$, then S is a union of $k+1$ and no fewer convex sets.

Even if we impose the additional restriction that Q not be contained in the union of the $(d-2)$-faces of P_0 for any polytope P_0 properly contained in P, examples show that a decomposition for S may require more than $\mathrm{c}(P')$ convex sets.

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